Dynamics of Belief Theoretic Agent Opinions Under Bounded Confidence

Ranga Dabarera, Rafael C. Núñez, Kamal Premaratne, and Manohar N. Murthi
Department of Electrical and Computer Engineering, University of Miami,
Coral Gables, Florida 33146 USA
ranga@miami.edu, nunez@umiami.edu, {kamal, mmurthi}@miami.edu

***Abstract*—Soft evidence sources play a critical role in social networks and similar settings, where subjective evidence and opinions are the norm. Study of opinion dynamics (including consensus and cluster formation) in these scenarios requires agent models that can capture the types of uncertainties and nuances characteristic of soft evidence (human-generated input, subjective evidence, etc.). To address the corresponding challenges, we employ a Dempster-Shafer (DS) belief theoretic agent model to explore opinion dynamics under bounded confidence. The proposed model further captures the notions of global affinity and the nature of persuasion of agents in social judgement theory. The paper develops several new results and these results regarding formation of clusters and consensus of agent opinions are verified with the aid of several numerical studies accompanied by bifurcation diagrams.**

***Index Terms*—opinion dynamics, bounded confidence, belief theory, consensus, cluster formation.**

I. INTRODUCTION

Collective behavior of multi-agent systems, where the behavior is determined by local interactions between neighboring agents, finds application in many areas of interest. In distributed control problems, the consensus state attempts to achieve a control objective [1]; in estimation problems, the agents attempt to estimate an underlying statistic of a signal [2]; within the context of fusion, the agents attempt to pool their evidence to arrive at a consensus decision [3]. To reach a consensus regarding a variable or some phenomenon of interest, the agents typically start with their own initial states and then iteratively exchange their states regarding their beliefs about the variable. Convergence analysis involves the study of such iterated belief revision processes among agents embedded in a networked environment.

With advances in sensing technology, information is now routinely extracted from large numbers of heterogeneous sources. These sources can include both *hard* sensors (i.e., conventional physics-based sensors) and *soft* sensors (i.e., human-based sources such as, expert opinions, subjective evidence, etc.). The importance of soft sources is particularly obvious in social network settings, which have become a dominant societal force. In a network where the agents may represent soft sources, consensus usually refers to a common agreement about an opinion of interest. Then, the questions of interest in consensus analysis are related to whether the agents will reach a common *consensus opinion* or whether they form *consensus opinion clusters* where each cluster consists of agents with similar opinions [4]. One can also look at the *leader-follower problem,* where agents follow other agents who possess stronger opinions [5], emergence of extremism, minority opinion spreading/survival, emergence of political parties, etc. [6].

A. Previous Work

Social Judgement Theory (SJT) discusses the basic psychological processes underlying the expression of attitudes and their modifiability through communication [7].

1) Boundedness: When a group of agents communicate between each other, a particular agent may adjust its opinion based mainly on the opinions of neighboring agents with similar opinions. In other words, an agent may be willing to update its opinion with the neighboring agent's opinion only if the 'distance' to that opinion is less than a certain bound of confidence ε. *Bounded confidence* refers to this phenomenon. The rationale for bounded confidence stems from the concept of *latitude of acceptance* in SJT.

In the sociophysics community, the *Hegselmann-Krause (HK) model* [8], [9] and the *Deffuant-Weisbuch (DW) model* [10]–[12] have attracted considerable attention for modeling real-valued opinions under bounded confidence [6]. Most of the work on HK and DW models have been carried out on a single opinion which is usually taken to be bounded in the range $[0, 1]$, where 0 and 1 represent the two strong extreme opinions whereas values in $(0, 1)$ represent weaker/stronger opinions towards the extremes.

2) Global Affinity: Fortunato, et al. [13] have considered vector opinions under bounded confidence. In such a scenario, the affinity of a single opinion may not capture the global affinity of the agents. For example, consider a situation where two agents A and B initially have different views on a particular political party, say T. So, if only this single opinion is considered, under a bounded confidence model, A and B may not exchange opinions. But, if A and B agree from a global point-of-view (e.g., they may have similar opinions about the other parties), under a bounded confidence model, they may exchange opinions (including opinions about T).

3) Nature of Persuasion: SJT further mentions that a receiving agent's *ego* involvement should also be taken into account when assessing opinion change [7]. Individuals with smaller ego involvement are easy to persuade. While such individuals tend to have a higher latitude of acceptance, nature of persuasion is a different notion than bounded confidence (which is accounted for via ε). For instance, one may find it difficult to persuade an agent possessing high ego in spite of it having neighboring agents with similar opinions. Hence, different opinion updating strategies may have to be implemented to account for nature of persuasion of each agent. In this work, we use two main opinion updating strategies named *cautious* and *receptive*.

B. Contributions

Most consensus studies employ an agent state which is modeled via real-valued vectors. However, such models are ill-suited to handle agent opinions because they cannot capture the types of uncertainties and the nuances characteristic of soft evidence (e.g., human-generated input/opinions, human domain expert opinions, etc.) in networked environments (e.g., in social network settings) adequately well. A better alternative is provided by imprecise probabilistic formalisms, such as Dempster-Shafer (DS) theory [14] where agent opinion can be captured via the DS theoretic (DST) notions of mass, belief, and plausibility. Consensus formation within such a DST framework has recently been explored in [15], [16].

The DST and Bayesian frameworks are closely related [17]. When the DST support values are restricted to only singletons, the DST notions of belief and plausibility yield probability mass functions (p.m.f.s). This allows for a smoother transition between DST and Bayesian notions. Our current work also employs the DST framework to model agent opinions. We also use *receptive* and *cautious* updating strategies as proposed in [18] and utilized in [15], [16] to account for nature of persuasion. However, this current work differs from [15], [16] because we now account for bounded confidence. As we show later, the receptive updating strategy with a larger bound of confidence can model agents who are open to opinions of others and thus are willing to change their opinions accordingly. On the other hand, the cautious updating strategy with a lower bound of confidence can capture the egocentric nature of an agent. Our work can also account for the concept of *forceful agents* as put forth in [19]. There are two ways to view forceful agents: *stubborn agents* and *community leaders/news media.* Stubbornness can be considered a form of egocentricity of agents. Community leaders can be modeled as cautious updating agents with a higher number of neighboring agents. They influence other agents in the community and change the opinion of receptive agents in a particular direction.

So, in essence, we analyze the consensus properties and group formation under bounded confidence of agents who are modeled within the DST framework. To the best of our knowledge, this current work constitutes the first instance of exploring the bounded confidence model within a DST opinion representation. This current work makes a significant advance towards bridging the gap between the DST framework for opinion dynamics and SJT (including bounded confidence, global affinity, and nature of persuasion).

Essential notions of DST are given in Section II. DST modeling of opinion dynamics appears in Section III. Consensus formation under bounded confidence within the DST framework is given in Section IV. In this preliminary work, we only address the case where the DST model which captures the agent opinion contains only singleton focal elements (i.e., the p.m.f. case). Section V gives the results of our numerical trials.

II. Preliminaries

A. Basic Notions in DS Theory

We use $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}_{\geq 0}$ to denote the integers, reals, and the non-negative reals, respectively.

In DS theory, the *frame of discernment (FoD)* refers to the discrete set $\Theta = \{\theta_1, \cdots, \theta_M\}$ of mutually exclusive and exhaustive propositions [14]. The cardinality $|\Theta| = M$ of Θ is the number of independent singleton propositions. A singleton proposition $\theta_i \in \Theta$ represents the lowest level of discernible information. The power set of the FoD $2^\Theta = \{A : A \subseteq \Theta\}$ denotes all the possible subsets of Θ. For $A \subseteq \Theta$, $\overline{A}$ denotes all singletons in Θ that are not in A.

1) Basic Belief Assignment (BBA): A *basic belief assignment (BBA)* or *mass assignment* is a mapping $m(\cdot) : 2^\Theta \mapsto [0, 1]$ such that $\sum_{A \subseteq \Theta} m(A) = 1$ and $m(\emptyset) = 0$. The BBA measures the "support" assigned to proposition $A \subseteq \Theta$. Propositions that receive non-zero mass are referred to as *focal elements.* The set of focal elements is the *core* $\mathcal{F}$. The triplet $\mathcal{E} = \{\Theta, \mathcal{F}, m\}$ is referred to as the *body of evidence (BoE).* We also use the notation $\widehat{\mathcal{F}} = \{A \subseteq \Theta : \text{Bl}(A) \neq 0\}$.

2) Notion of Ignorance: In DST, focal elements can be any singleton or composite (i.e., non-singleton) proposition. DST captures the notion of *ignorance* by allocating masses to composite propositions. For instance, the composite proposition $\{\theta_i\theta_j\}$, $\theta_i, \theta_j \in \Theta$, is a doubleton and the mass assignment $m(\theta_i\theta_j) > 0$ represents ignorance or lack of evidence to differentiate between the two constituent singletons. The state of complete ignorance can be easily captured via the *vacuous BBA* 1_Θ which has Θ as its only focal element, i.e., the mass assignment structure of the vacuous BBA is $m(A) = 1$ for $A = \Theta$ (and hence $m(A) = 0$ for $A \subset \Theta$).

A BBA is called *Bayesian* if each focal element is a singleton. For a Bayesian BBA, the BBA, belief, and plausibility, all reduce to a probability assignment.

3) Belief and Plausibility: Given a BoE, $\mathcal{E} \equiv \{\Theta, \mathcal{F}, m\}$, the *belief* function $\text{Bl} : 2^\Theta \mapsto [0, 1]$ is defined as $\text{Bl}(A) = \sum_{B \subseteq A} m(B)$. $\text{Bl}(A)$ represents the total belief that is committed to A without also being committed to its complement $\overline{A}$. The *plausibility* function $\text{Pl} : 2^\Theta \mapsto [0, 1]$ is defined as $\text{Pl}(A) = 1 - \text{Bl}(\overline{A})$. It corresponds to the total belief that does not contradict A. The *uncertainty* of A is $[\text{Bl}(A), \text{Pl}(A)]$.

4) DS Theoretic Conditionals: Of the various notions of DST conditionals abound in the literature, the Fagin-Halpern (FH) conditional offers a unique probabilistic interpretation and hence a natural transition to the Bayesian conditional notion [17], [20]. The extensive study in [21] identifies several attractive properties of the FH conditionals including its equivalence to other popular notions of DST conditionals.

Definition 1 (FH Conditionals)**.** *For the BoE* $\mathcal{E} = \{\Theta, \mathcal{F}, m\}$ *and* $A \subseteq \Theta$ *s.t.* $A \in \widehat{\mathcal{F}}$, *the* conditional belief $\text{Bl}(B|A) : 2^\Theta \mapsto$

$[0,1]$ *and* conditional plausibility $\text{Pl}(B|A) : 2^\Theta \mapsto [0,1]$ *of* B *given* A *are*

$$\text{Bl}(B|A) = \frac{\text{Bl}(A \cap B)}{\text{Bl}(A \cap B) + \text{Pl}(A \cap \overline{B})};$$
$$\text{Pl}(B|A) = \frac{\text{Pl}(A \cap B)}{\text{Pl}(A \cap B) + \text{Bl}(A \cap \overline{B})}.$$

The conditional core theorem [22] can be utilized to directly identify the conditional focal elements to improve computational performance when applying FH conditionals.

B. Conditional Update Equation (CUE)

The conditional update equation (CUE) [18], [23] offers a strategy to update evidence from different BoEs $\mathcal{E}_j \equiv \{\Theta_j, \mathcal{F}_j, m_j\}$, $j = 1, 2, \ldots, N$, to arrive at a new updated BoE $\mathcal{E} = \{\Theta, \mathcal{F}, m\}$. Without loss of generality, let us consider updating the BoE $\mathcal{E}_1$ with the evidence in $\mathcal{E}_i$, $i = 2, 3, \ldots, N$, to generate the BoE $\mathcal{E}$. In the CUE, the updated belief of an arbitrary proposition B in the BoE $\mathcal{E}$ is given by

$$\text{Bl}_1(B)_{(k+1)} = \alpha_1 \text{Bl}_1(B)_k + \sum_{i=2}^{N} \sum_{A \in \widehat{\mathcal{F}}_i} \beta_{1,i}(A)\, \text{Bl}_i(B|A)_k. \tag{1}$$

We denote this as $\mathcal{E} \equiv \mathcal{E}_1 \lhd (\mathcal{E}_2 \bowtie \cdots \bowtie \mathcal{E}_N)$. Here, α_1 and $\beta_{1,i}$ are non-negative parameters that satisfy $\alpha_1 + \sum_{i=2}^{N} \sum_{A \in \widehat{\mathcal{F}}_i} \beta_{1,i}(A) = 1$.

1) Selection of the CUE Parameters: The work in [18] provides different strategies for the selection of the CUE parameters. The parameters α_1 determines the flexibility of accepting updates from neighbors. The lower the value of α_1, the higher the flexibility of the CUE update towards changes. The parameters $\beta_{1,i}$ weigh the updates from incoming evidence. These parameters can be selected to be proportional to the support available for the corresponding focal element from $\mathcal{E}_1$ or $\mathcal{E}_i$. This generates two strategies which are referred to as *cautious* and *receptive* strategies, respectively [18].

III. DST Modeling of Opinion Dynamics

Let us consider N agents with BoEs $\mathbf{A} = \{\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_N\}$ embedded in a directed graph $G[k] = (\mathbf{A}, E[k])$ at the discrete-time instant $k \in \mathbb{N}$. Each node in $G[k]$ represents an agent; a directed edge $e_{ij} \in E[k]$ represents unidirectional information exchange link from agent j to agent i (i.e., agent i can receive information from agent j), and e_{ij} is an element in the adjacency matrix $E[k]$ of graph $G[k]$.

The opinion of agent $i \in \{1, 2, \cdots, N\}$ at discrete-time index $k \in \mathbb{N}$ is modeled via the BBA $m^k_{\Theta_i}$. We assume that the FoDs associated with the agents are identical and equal to Θ. So, henceforth, we will denote $m^k_{\Theta_i}$ simply by m^k_i. Given an arbitrary $B \subseteq \Theta$, the BBA that agent i associates with B at time index k is $m_i(B)_k$. The *opinion profile* of $B \subseteq \Theta$ at time index k is the size N vector $\mathbf{m}(B)_k = [m_1(B)_k, \ldots, m_N(B)_k]^T$. The initial state of the opinion profile of B is $\mathbf{m}(B)_0$.

A. Bounded Confidence

The process of updating an agent's opinion is as follows. First, agent i judges how close the opinion of its neighbors j are. Then, agent i updates $\mathcal{E}_i$ in response to the opinion $\mathcal{E}_j$ if $\mathcal{E}_j$ is within the latitude of acceptance (bound of confidence) ε_i, i.e., $0 \leq \|\mathcal{E}_i - \mathcal{E}_j\| \leq \varepsilon_i$ where $\|\cdot\|$ denotes any valid norm.

In the DW model, the threshold ε is referred to as an *openness character* [10]. Another interpretation views ε as an *uncertainty,* i.e., if agent i possesses an opinion with some degree of uncertainty ε_i, then agent i does not care about the views of other agents outside its uncertainty range. Depending on the bound of confidence, agent-based models can be considered as either *heterogeneous* or *homogeneous*. A heterogeneous model is where each agent possesses a unique bound of confidence ε_i, distinct from that of its neighbors. In a homogeneous model, all bounds of confidence are assumed to be equal, i.e., $\varepsilon_1 = \cdots = \varepsilon_N \equiv \varepsilon$.

Note that a model based on bounded confidence requires a definition of "closeness" and the selection of a suitable measure of closeness is crucial for opinion updating under bounded confidence. If agent opinions are represented via real numbers or singletons (e.g., each agent opinion is represented by a real-valued vector), then Euclidean norm could serve as a measure of closeness of opinions. In our case, agent opinions are captured via DST BoEs, hence the distance measure has to account for the closeness of BoEs. While noting that other DST distance measures may serve as valid candidates, we use the DST distance measure in [24], [25].

Definition [24] The distance between the two BoEs $\mathcal{E}_i = \{\Theta, \mathcal{F}_i, m_i\}$ and $\mathcal{E}_j = \{\Theta, \mathcal{F}_j, m_j\}$, where $|\Theta| = M$, is

$$\|\mathcal{E}_i - \mathcal{E}_j\| = \sqrt{\frac{1}{2}(\boldsymbol{m}_i - \boldsymbol{m}_j)^T \mathcal{D}\,(\boldsymbol{m}_i - \boldsymbol{m}_j)},$$

where $\boldsymbol{m}_i$ and $\boldsymbol{m}_j$ are $2^M \times 1$ column vectors formed from the mass assignments, $\mathcal{D}$ is a $2^M \times 2^M$ matrix with elements $\mathtt{d}_{mn} = |A_m \cap A_n|/|A_m \cup A_n|$, $A_m, A_n \subseteq \Theta$, with $|\emptyset \cap \emptyset|/|\emptyset \cup \emptyset| \equiv 0$.

B. Opinion Updating

We assume that each agent updates its opinion in accordance with the CUE in (1). Our opinion update model adopts the receptive and cautious updating strategies proposed in [18]. Essentially, the selection of parameters $\beta_{1,i}$ in (1) determines the cautiousness or the receptiveness of an agent as follows:

(i) Receptive updating: Select $\beta_{1,i}(A)[\cdot] = K_{\Theta_i} m_{\Theta_i}(A)[\cdot]$.
(ii) Cautious updating: Select $\beta_{1,i}(A)[\cdot] = K_{\Theta_1} m_{\Theta_1}(A)[\cdot]$.

Here, K_{Θ_i} and K_{Θ_1} are constants.

According to the number of cautious agents present, we have the following cases to consider:

1) **All receptive agents**: This is the most common community or network agent type that has been discussed in literature [6], [9], [10].
2) **One cautious agent**: Here, the group of agents contain all receptive agents except one cautious updating agent (e.g., the scenarios considered in typical leader-follower models [2], the recent work in [15], [16]).

3) **Multiple cautious agents:** Here, there are more than one cautious updating agents, generally with different initial opinions. To our knowledge, this case has not been addressed previously within the DST framework.

C. Consensus and Cluster Formation

Consensus in a DST environment has been explored in [15], [16], where the theory of paracontractions has been utilized for consensus analysis. Our analysis in Section IV is based on, and extends, the work in [4], [8] and provides sufficient conditions for consensus/cluster formation.

IV. Consensus Formation Under Bounded Confidence

Consider our N BoEs $\{\mathcal{E}_1, \ldots, \mathcal{E}_N\}$ defined on the same FoD $\Theta = \{\theta_i, \ldots, \theta_M\}$. Note that $m_i(\theta_w)_k$ denotes the opinion on singleton $\theta_w \in \Theta$ of agent $i \in \{1, \ldots, N\}$ at discrete-time $k \in \mathbb{N}$. The vector $\mathbf{m}(\theta_w)_k = [m_1(\theta_w)_k, \ldots, m_N(\theta_w)_k]^T \in \mathbb{R}^N_{\geq 0}$ is the opinion profile of singleton opinion θ_w at time k.

Suppose agent i updates $\mathcal{E}_i$ by taking into account the opinions of all agents j whose BoEs $\mathcal{E}_j$ lie within the distance ε_i from agent i's own opinion, i.e., $\|\mathcal{E}_i - \mathcal{E}_j\| \leq \varepsilon_i$. Here, $\varepsilon_i > 0$ is the bound of confidence of agent i based on the selected norm. The opinion update is modeled via the CUE in (1) which, in terms of masses, can be expressed as

$$m_i(B)_{(k+1)} = \alpha_{i,k} m_i(B)_k + \sum_{j \neq i} \sum_{A \in \mathcal{F}_{j,k}} \beta_{ij}(A)_k m_j(B|A)_k, \tag{2}$$

where $i, j \in \{1, \ldots, N\}$, $\alpha_{i,k}$ and $\beta_{ij}(\cdot)_k$ are non-negative real numbers satisfying

$$\alpha_{i,k} + \sum_{j \neq i} \sum_{A \in \mathcal{F}_{j,k}} \beta_{ij}(A)_k = 1. \tag{3}$$

For mathematical convenience, we assume a homogeneous bounded confidence model, i.e., all agents are taken to have the same confidence range $\varepsilon_i \equiv \varepsilon$. In addition, we also assume a static network (in the sense that the communication links are static). The analytical results below apply to the case where the DST models contain only singleton focal elements (i.e., the p.m.f. case).

Definition 2. *For agent $i \in \{1, \ldots, N\}$, the set of neighborhood agents at discrete-time instance k is*

$$I_k(i) = \{j = 1, \ldots, N : \|\mathcal{E}_i - \mathcal{E}_j\| \leq \varepsilon\}.$$

The number of neighbor agents of agent i is $\tau_{i,k} = |I_k(i)|$.

For the case of DST models that has only singleton focal elements, one can show that the opinion update can be written as the following discrete-time dynamical system:

$$\mathbf{m}(\theta_w)_{k+1} = \mathbf{A}_k \mathbf{m}(\theta_w)_k, \tag{4}$$

where the initial conditions are denoted as $\mathbf{m}(\theta_w)_0 \in (\mathbb{R}_{\geq 0})^N$ and the $N \times N$ matrix $\mathbf{A}_k = \{a_{ij,k}\}$ is defined as

$$a_{ij,k} = \begin{cases} \alpha_{i,k}, & \text{for } i = j; \\ \gamma_{i,k}, & \text{for } j \in I_x(i); \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

Here, $\gamma_{i,k} = (1 - \alpha_{i,k})/\tau_{i,k}$.

We also use the following notion [4], [8]:

Definition 3. *The range $\mathcal{R}$ of a singleton opinion profile θ_w at discrete-time k is*

$$\mathcal{R}(\mathbf{m}(\theta_w)_k) = \max_{1 \leq i,j \leq N} (m_i(\theta_w)_k - m_j(\theta_w)_k), \; \theta_w \in \Theta.$$

In [8], sufficient conditions for consensus have been given using a certain ε-profile. However, this previous work only considers real-valued agent opinions. Here, we extend the result to p.m.f.s under certain assumptions on DST BoEs.

Similar to the real-valued opinions case in [4], [8], let us assume that our BoEs can be arranged as an *ordered list* according to the distance relative to a particular 'reference' BoE. Without loss of generality, we take $\mathcal{E}_1$ as the reference BoE and re-label all the BoEs as $\mathcal{E}_1, \ldots, \mathcal{E}_N$ so that $\|\mathcal{E}_2 - \mathcal{E}_1\| \leq \ldots \leq \|\mathcal{E}_N - \mathcal{E}_1\|$. If this order does not change when the opinions of agents get updated each time, then it is called an *order preserving* arrangement. Furthermore, if each singleton opinion profile can be arranged in the same order as the BoE indices as ascending or descending mass values, then it is referred to as a *strictly order preserved* arrangement.

Definition 4. *Under the strictly order preserved arrangement, the BoE setting is said to have a ε-chain if each agent has distance to its neighbors less than the bound of confidence ε such that*

$$\|\mathcal{E}_{i+1} - \mathcal{E}_i\| \leq \varepsilon, \; \forall i = 1, \ldots, N-1.$$

A. All Receptive Agents

We first explore consensus when all agents are receptive. For this purpose, we consider a strictly order preserved arrangement and analyze fixed points of the agent opinions. For such an arrangement, the diagonal and off-diagonal entries of the transition matrix $\mathbf{A}_k$ are positive (Proposition 3 in [4]). Furthermore, a product of $(N-1)$ such matrices, say $\mathbf{B}_\zeta = \{b_{ij,\zeta}\}$, is positive and given by

$$\mathbf{B}_\zeta = \mathbf{A}_{(N-1)(\zeta+1)-1} \cdots \mathbf{A}_{(N-1)\zeta} > 0, \tag{6}$$

i.e., $b_{ij,\zeta} > 0$, for all i, j, and $\zeta \in \mathbb{N}$.

For all singleton opinion profiles $\mathbf{m}(\theta_w)$, $\theta_w \in \Theta$, the updates at discrete-time $(k+1)$, such that $k \in (N-1) \cdot \mathbb{N}$, can be given as

$$\mathbf{m}(\theta_w)_{k+1} = \mathbf{B}_\zeta \mathbf{B}_{\zeta-1} \cdots \mathbf{B}_0 \mathbf{m}(\theta_w)_0, \tag{7}$$

where $\zeta = (k+1)/(N-1) - 1$.

From [8], for a non-negative row stochastic matrix $\mathbf{C}$, we have the following result:

Lemma 1. *[8] When a matrix $\mathbf{C}$ is row stochastic, then, for all $\mathbf{m}(\theta_w) \in \mathbb{R}^N_{\geq 0}$,*

$$\mathcal{R}(\mathbf{C}\mathbf{m}(\theta_w)_k) \leq \left(1 - \min_{1 \leq i,j \leq N} \sum_{\ell=1}^{N} \min\{a_{i\ell,k}, a_{j\ell,k}\}\right) \mathcal{R}(\mathbf{m}(\theta_w)_k). \tag{8}$$

As the matrices $\mathbf{B}_\zeta$s are positive row-stochastic, from Lemma 1 we have

Lemma 2. *For all positive row-stochastic matrices* $\mathbf{B}_\zeta$ *s,*

$$\mathcal{R}(\mathbf{B}_\zeta \mathbf{m}(\theta_w)_{k+1}) \leq \lambda \mathcal{R}(\mathbf{m}(\theta_w)_{k+1}), \text{ for some } \lambda < 1.$$

From Lemma 2, we know that $\mathcal{R}(\mathbf{B}_\zeta \mathbf{m}(\theta_w)_{k+1}) < \mathcal{R}(\mathbf{m}(\theta_w)_{k+1})$. Hence, from Corollary 7 in [4], we can form a sequence $(\mathcal{R}(\mathbf{m}(\theta_w)_{\tilde{k}}))_{\tilde{k}}$ that converges to 0. Due to the row stochasticity of $\mathbf{A}_k$, from Lemma 1, we can show that the sequence $(\mathcal{R}(\mathbf{m}(\theta_w)_k))_k$ is monotonically decreasing and indeed a Cauchy sequence. But, $(\mathcal{R}(\mathbf{m}(\theta_w)_{\tilde{k}}))_{\tilde{k}}$ is a subsequence of the sequence $(\mathcal{R}(\mathbf{m}(\theta_w)_k))_k$. Hence the sequence $(\mathcal{R}(\mathbf{m}(\theta_w)_k))_k$ converges to 0.

When $(\mathcal{R}(\mathbf{m}(\theta_w)_k))_k \to 0, \forall \theta_w \in \Theta$, we reach a consensus. In summary, we have shown that consensus will be reached for a strictly order preserved arrangement of BoEs with an ε-chain. However, when a 'crack' appears in the ε-chain, the agents get divided into independent groups, and the above result applies to each subgroup yielding a fixed point with separate cluster points for each independent group.

B. One Cautious Agent

When a cautious agent, $\mathcal{E}_{C1}$, $C1 \in \{1, \ldots, N\}$, is present with all the others being receptive agents, the transition matrix $\mathbf{A}_k$ at time k can be given as in

$$\mathbf{A}_k = \begin{pmatrix} P_k & \mathbf{u}_k & S_k \\ \mathbf{0}^T & 1 & \mathbf{0}^T \\ R_k & \mathbf{v}_k & Q_k \end{pmatrix}, \tag{9}$$

where

$$\begin{aligned} P_k, Q_k, R_k, S_k &= \text{square matrices of appropriate size;} \\ \mathbf{u}_k, \mathbf{v}_k &= \text{vectors of appropriate size;} \\ \mathbf{0} &= \text{zero vectors of appropriate size.} \end{aligned}$$

The matrix in (9) is row-stochastic. Hence, Lemma 1 can be applied. Further, for the all singleton scenario, as the opinion of the cautious agent does not change, it can be seen that

$$\begin{aligned} \min_{y \in I_k(C1)} [m_y(\theta_w)_{k+1}] &\leq m_{C1}(\theta_w)_{k+1} \\ &\leq \max_{z \in I_k(C1)} [m_z(\theta_w)_{k+1}], \end{aligned} \tag{10}$$

for all $y, z \in I_k(C1)$, $\theta_w \in \Theta$. Hence the neighboring receptive agents of cautious agent $C1$ converge to the opinion of the leader, viz., $C1$. When the conditions for the ε-chain are satisfied, a consensus will be reached. The fixed point is given by the masses of the cautious agent's singleton opinions. Again, when a crack appears in the ε-chain, agents get divided into independent groups. However, the group that contains the cautious agent converges to the cautious agent's opinion.

C. Multiple Cautious Agents

If there are more than one cautious agent, clearly there will be no consensus (unless off course all the cautious agents have the same opinion). In a strictly order preserved arrangement with an ε-chain, the receptive agents who get updated from only one cautious agent converge to that particular cautious agent's opinion. The analysis is similar to the scenario with one cautious agent (see Section IV-B).

When there is more than one cautious agent in the bound of confidence of a receptive agent, two possibilities could happen.

(i) If the receptive agent continues to have more than one cautious agent for iterations to come, then it will converge to a fixed point. The fixed point is in the convex hull of the set of points corresponding to neighboring cautious agents.

(ii) Even if the receptive agent's neighborhood initially contains more than one cautious agent, this neighborhood could later contain fewer cautious agents. This situation can be addressed as in IV-B or item (i) above.

V. Results and Discussion

We now study the formation of consensus/clusters based on the proposed methodology. We explore all the three scenarios: all receptive agents, one cautious agent, and multiple cautious agents. The results of consensus/cluster-formation are shown with the aid of bifurcation diagrams which depict the state of cluster formation/consensus in the limit density versus the bound of confidence ε [6], [26].

All the scenarios use 100 agents, each having the identical FoD $\Theta = \{\theta_1, \theta_2, \theta_3\}$. An agent i's opinion is represented via a DST model with focal elements restricted to be singleton propositions, i.e., the opinion model is essentially a p.m.f. with probabilities $\{m_i(\theta_1), m_i(\theta_2), m_i(\theta_3)\}$, where $\sum_{j=1}^{3} m_i(\theta_j) = 1$.

A. All Receptive Agents

1) Uniformly Distributed Case: Most studies on real-valued opinion dynamics (with agents having a single opinion in the range $[0, 1]$) have used random and uniformly distributed initial opinion profiles or initial densities that are uniformly distributed in the opinion space [6]. In accordance with these previous works, we have conducted a trial experiment with 100 receptive agents, assuming that the initial DST mass of the opinion on θ_1 is uniformly distributed in the range $[0, 1]$, i.e., $m_i(\theta_1) = \mathcal{U}(0, 1)$, $i \in \{1, \ldots, 100\}$. The remaining mass is equally distributed among the other singletons θ_2 and θ_3. Fig. 1 shows the corresponding bifurcation diagram with respect to θ_1. It can be seen that, for smaller values of ε (approximately < 0.12), no consensus is formed. Indeed, the lower the value of ε, the higher the number of clusters. The cluster formation at $\varepsilon = 0.1$ and consensus at $\varepsilon = 0.3$ appear in Figs 2a and 2b, respectively. The analysis in Section IV-A directly applies to this scenario, where an ε-chain is formed for $\varepsilon > 0.12$.

2) Dirichlet Distribution Case: Further experiments were carried out by sampling opinions from a Dirichlet distribution [27], i.e., $m_i(\theta_1, \theta_2, \theta_3; 2, 2, 2) = Dir(2, 2, 2)$, $i \in \{1, \ldots, 100\}$. The symmetric Dirichlet distribution corresponds to the case having no prior information to favor one singleton over the other. The symmetric Dirichlet distribution with concentration parameter equal to one is equivalent to a uniform distribution over the open standard 2-simplex. The parameters $(2, 2, 2)$ in the Dirichlet distribution enforce a symmetric and dense distribution with a centered mode. Fig. 3 shows the bifurcation diagram with all receptive agents. It can be seen that consensus is reached when $\varepsilon > 0.15$

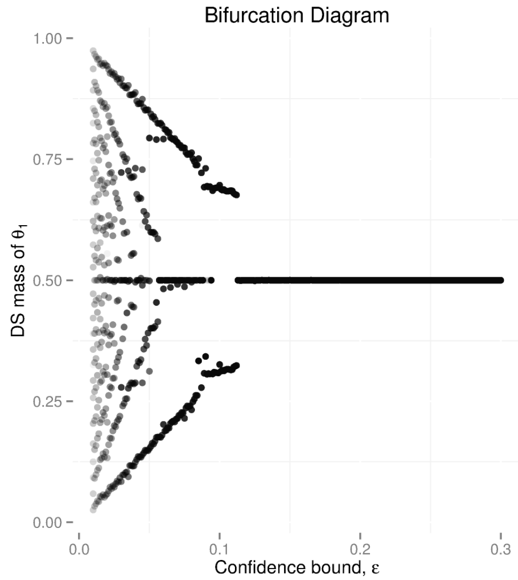
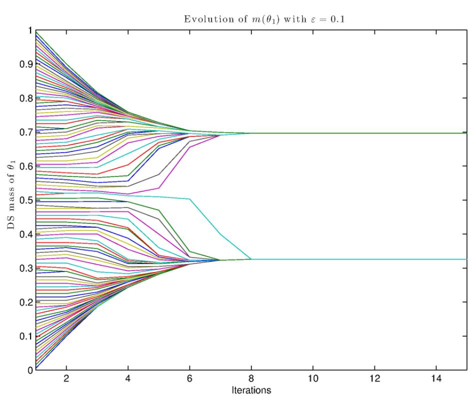
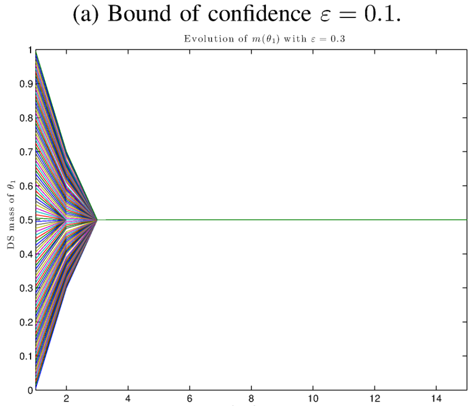


Fig. 1: All receptive agents case/mass for θ_1 sampled from a uniform distribution: bifurcation diagram for θ_1. The mass values of singleton θ_1 of agents in the limit density are given for different bound of confidence values ε. The intensity of each point corresponds to the denseness of the representing cluster.



(a) Bound of confidence $\varepsilon = 0.1$.



(b) Bound of confidence $\varepsilon = 0.3$

Fig. 2: All receptive agents case/mass for θ_1 sampled from a uniform distribution: evolution of opinion profile of θ_1.

(approximately). Fig. 4 shows that the minimum bound of confidence ε required for consensus gets lower as the number of agents increases. This is to be expected because, as the number of agents increases, it is easier to make an ε-chain with a lower bound of confidence.

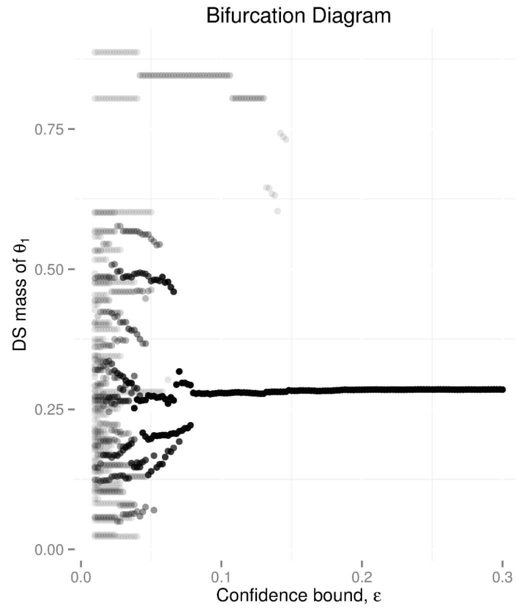


Fig. 3: All receptive agents case/masses of all singletons sampled from $Dir(2, 2, 2)$: bifurcation diagram for θ_1.

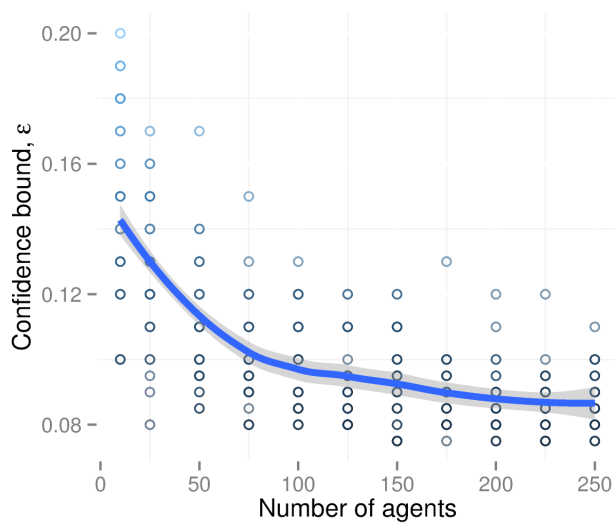


Fig. 4: All receptive agents case/masses of all singletons sampled from $Dir(2, 2, 2)$: minimum bound of confidence required for consensus versus the number of receptive agents (solid line: estimated minimum bound of confidence, shaded: standard error).

B. One Cautious Agent

A test with one cautious agent and 99 receptive agents was carried by selecting the initial mass assignment of the cautious agent as, $m_{C1}(\theta_1) = 0.50$, $m_{C1}(\theta_2) = 0.25$ and $m_{C1}(\theta_3) = 0.25$. Masses of the receptive agents were sampled from $Dir(2, 2, 2)$. The bifurcation diagram with respect to θ_1 is given in Fig. 5, where consensus is reached when $\varepsilon > 0.13$ (approximately). As expected from the analysis in Section IV-B, the cautious agent acts as an opinion leader and guides the consensus by influencing all the other receptive agents to converge to the cautious BoE $\mathcal{E}_{C1}$.

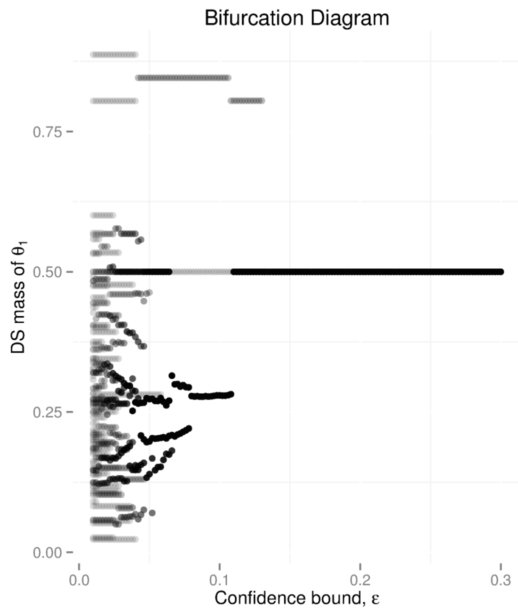


Fig. 5: One cautious agent case/masses of receptive agents sampled from $Dir(2, 2, 2)$/masses of cautious agent $\{m_{C1}(\theta_1), m_{C1}(\theta_2), m_{C1}(\theta_3)\} = \{0.50, 0.25, 0.25\}$: bifurcation diagram for θ_1.

C. Multiple (Two) Cautious Agents

Fig. 6 gives the bifurcation diagram for the scenario with two cautious agents $C1, C2$ and 98 receptive agents. The mass assignment for the initial state of cautious agents are $m_{C1}(\theta_1) = 0.75$, $m_{C1}(\theta_2) = 0.125$, $m_{C1}(\theta_3) = 0.125$, and $m_{C2}(\theta_1) = 0.25$, $m_{C2}(\theta_2) = 0.375$ and $m_{C2}(\theta_3) = 0.375$. The masses of the opinions of receptive agents were sampled from $Dir(2, 2, 2)$. As the 'stubborn' opinion leaders carry different opinions, we will not see any consensus in this scenario (see discussion in Section IV-C). The minimum number of two clusters can be observed for $0.08 < \varepsilon < 0.23$ (approximately), where receptive agents are influenced by the closest opinion leaders and cling to the closest group (see item (ii) of Section IV-C). Note that the singleton opinions on θ_1 have been clustered to two groups with masses belonging to either $m_{C1}(\theta_1)$ or $m_{C2}(\theta_1)$; other singletons behave similarly. Further, since the majority of the receptive agents were initially closer to the cautious agent $C2$, as can be seen from the higher intensity line in Fig. 6, the group formed under $C2$ has a higher agent density compared to the group with $C1$. However, for $\varepsilon > 0.23$, the number of clusters is fixed at 3 (see item (ii) of Section IV-C). When the receptive agents have a higher bound of confidence, they get influenced by both opinion leaders, thus forming a group where the majority of the group has opinions in the convex hull of the leader opinions.

We also make another observation. In the presence of cautious agents, the number of iterations required before reaching a fixed point is higher compared to that of a scenario with all receptive agents. Fig. 7 shows that, in the presence of cautious agents, it requires about 200 iterations to reach a fixed point, whereas in all receptive agents case Fig. 2, it has taken less than 10 iterations to reach a fixed point.

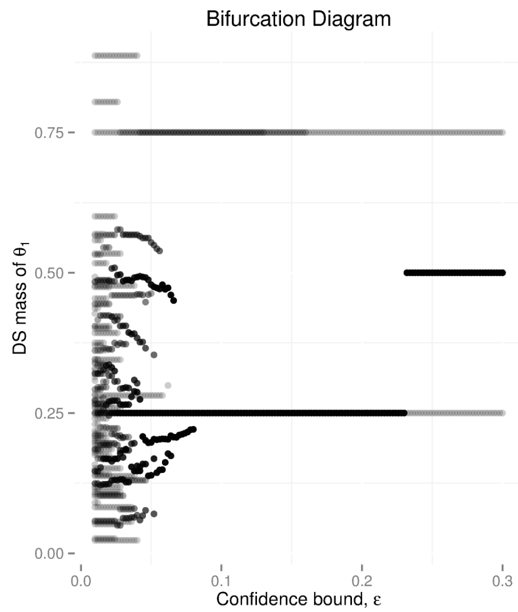
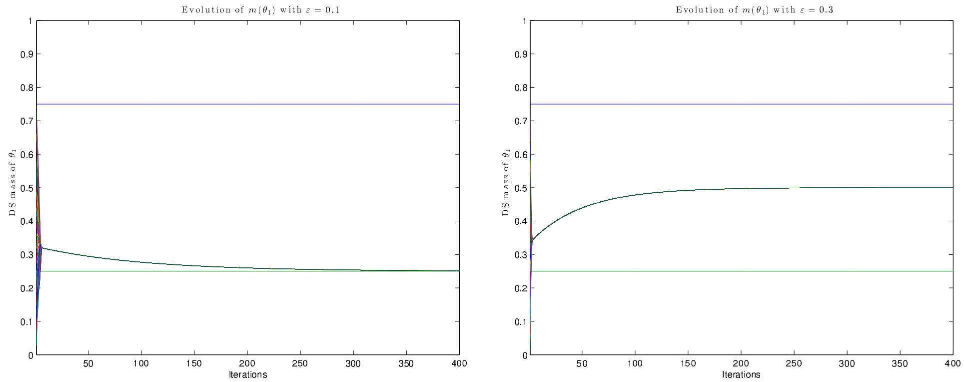


Fig. 6: Two cautious agents case/masses of receptive agents sampled from $Dir(2, 2, 2)$/masses of cautious agents are $\{m_{C1}(\theta_1), m_{C1}(\theta_2), m_{C1}(\theta_3)\} = \{0.75, 0.125, 0.125\}$, $\{m_{C2}(\theta_1), m_{C2}(\theta_2), m_{C2}(\theta_3)\} = \{0.25, 0.375, 0.375\}$: bifurcation diagram for θ_1.

VI. Conclusion

In this paper, we have utilized the DST framework to capture agent opinion and address opinion dynamics while taking into account aspects of bounded confidence, global affinity, and the nature of persuasion. Analytical, as well as numerical results, have been given assuming a homogeneous bounded confidence model. The global affinity of agents has been represented by considering multiple singletons in the DST mass function, while the nature of persuasion has been modeled with cautious and receptive updating strategies. The bifurcation diagrams for trials with all receptive agents and one cautious agent show that a consensus is reached after a threshold value of the bound of confidence of agents. All other scenarios, including trials with multiple cautious agents,



(a) $\varepsilon = 0.1$ (b) $\varepsilon = 0.3$

Fig. 7: Two cautious agents case/masses of receptive agents sampled from $Dir(2, 2, 2)$/masses of cautious agents are $\{m_{C1}(\theta_1), m_{C1}(\theta_2), m_{C1}(\theta_3)\} = \{0.75, 0.125, 0.125\}$, $\{m_{C2}(\theta_1), m_{C2}(\theta_2), m_{C2}(\theta_3)\} = \{0.25, 0.375, 0.375\}$: evolution of opinion profile of θ_1.

demonstrate cluster formation among opinions of agents. The proposed DST framework can be used to model opinion dynamics of multi-agent systems where soft data play a critical role (e.g., in social network settings).

ACKNOWLEDGEMENT

This work is based on research supported by the US Office of Naval Research (ONR) via grants #N00014-10-1-0140, #N00014-11-1-0493, and the US National Science Foundation (NSF) via grant #1038257.

REFERENCES

[1] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proc. IEEE*, vol. 95, no. 1, pp. 215–233, Jan. 2007.

[2] W. Ren, R. W. Beard, E. M. Atkins, and A. G. Theory, "A survey of consensus problems in multi-agent coordination," in *Amer. Cont. Conf. (ACC)*, pp. 1859–1864, 2005.

[3] L. Xiao, S. Boyd, and S. Lall, "A scheme for robust distributed sensor fusion based on average consensus," in *Int. Symp. on Inf. Proc. in Sensor Netw. (IPSN)*, pp. 63 – 70, Apr. 2005.

[4] J. Dittmer, "Consensus formation under bounded confidence," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 47, pp. 4615–4621, 2001.

[5] X. Lu, R. Lu, S. Chen, and J. Lu, "Finite-time distributed tracking control for multi-agent systems with a virtual leader," *IEEE Trans. Circ. and Syst. I: Regular Papers*, vol. 60, no. 2, pp. 352–362, Feb. 2013.

[6] J. Lorenz, "Continuous opinion dynamics under bounded confidence: A survey," *Int. J. of Modern Phys. C*,vol. 18,no. 12, pp. 1819–1838, Jul. 2007.

[7] M. Sherif and C. Hovland, *Social Judgment: Assimilation and Contrast Effects in Communication and Attitude Change*, ser. Yale Studies in Attitude and Communication. Greenwood Pub. Group, Inc., 1980.

[8] U. Krause, "A discrete nonlinear and non-autonomous model of consensus formation," *Comm. Diff. Equations*, no. 1, pp. 227–236, 2000.

[9] R. Hegselmann and U. Krause, "Opinion dynamics and bounded confidence models, analysis and simulation," *J. of Art. Societies and Social Simulation*, vol. 5,no. 3, Jun. 2002.

[10] G. Deffuant, D. Neau, F. Amblard, and G. Weisbuch, "Mixing beliefs among interacting agents," *Adv. in Complex Systems*, vol. 03, no. 04, pp. 87–98, Jan. 2000.

[11] G. Weisbuch, G. Deffuant, F. Amblard, and J.-P. Nadal, "Meet, discuss, and segregate!" *Complexity*, vol. 7, no. 3, pp. 55–63, Jan. 2002.

[12] G. Weisbuch, "Bounded confidence and social networks," *The European Physical Journal B - Condensed Matter*, vol. 38, no. 2, pp. 339–343, Mar. 2004.

[13] S. Fortunato, V. Latora, A. Pluchino, and A. Rapisarda, "Vector Opinion dynamics in a bounded confidence consensus model," *Int. J. of Modern Phys. C*, vol. 16, no. 10, pp. 1535–1551, Oct. 2005.

[14] G. Shafer, *A Mathematical Theory of Evidence*. Princeton Univ. Press, 1976.

[15] T. L. Wickramrathne, K. Premaratne, and M. N. Murthiy, "Convergence analysis of consensus belief functions within asynchronous ad-hoc fusion networks," in *IEEE Int. Conf. on Acoustics, Speech and Sig. Proc. (ICASSP)*. pp. 3612–3616, May 2013.

[16] T. L. Wickramarathne, K. Premaratne, M. N. Murthi, and N. Chawla, "Convergence analysis of iterated belief revision in complex fusion environments," *IEEE J. of Selected Topics in Sig. Proc., Special Issue on Sig. Proc. for Social Netw.*, Aug. 2014, to appear.

[17] R. Fagin and J. Y. Halpern, "A new approach to updating beliefs," in *Proc. Conf. on Uncertainty in Art. Intell. (UAI)*, P. P. Bonissone, M. Henrion, L. N. Kanal, and J. F. Lemmer, Eds. New York, NY: Elsevier Science,pp. 347–374, 1991.

[18] K. Premaratne, M. N. Murthi, J. Zhang, M. Scheutz, and P. H. Bauer, "A Dempster-Shafer theoretic conditional approach to evidence updating for fusion of hard and soft data," in *Proc. Int. Conf. on Inf. Fusion (FUSION)*, Seattle, WA, pp. 2122–2129, Jul. 2009.

[19] D. Acemoglu and A. Ozdaglar, "Opinion dynamics and learning in social networks," *Dynamic Games and Appl.*, vol. 1, no. 1,pp. 3–9, 2011.

[20] T. L. Wickramarathne, K. Premaratne, and M. N. Murthi, "Towards efficient computation of the Dempster-Shafer belief theoretic conditionals," *IEEE Trans. on Cyber.*, vol. 43, no. 2, pp. 712–724, Apr. 2013.

[21] D. Denneberg, "Conditioning (updating) non-additive measures," *Annals of Operations Research*, vol. 52, no. 1, pp. 21–42, Mar. 1994.

[22] T. L. Wickramarathne, K. Premaratne, and M. N. Murthi, "Focal elements generated by the Dempster-Shafer theoretic conditionals: A complete characterization," in *Proc. Int. Conf. on Inf. Fusion (FUSION)*, Edinburgh, UK, pp. 1–8, 2010.

[23] K. Premaratne, D. A. Dewasurendra, and P. H. Bauer, "Evidence combination in an environment with heterogeneous sources," *IEEE Trans. on Syst., Man, and Cyber. - A: Syst. and Humans*, vol. 37, no. 3, pp. 298–309, 2007.

[24] A.-L. Jousselme, D. Grenier, and E. Bossé, "A new distance between two bodies of evidence," *Inf. Fusion*, vol. 2, no. 2, pp. 91–101, Jun. 2001.

[25] A. Martin, A. L. Jousselme, and C. Osswald, "Conflict measure for the discounting operation on belief functions," *Inf. Fusion*, pp. 1–8, Jun. 2008.

[26] E. Ben-Naim, P. Krapivsky, and S. Redner, "Bifurcations and patterns in compromise processes," *Physica D: Nonlinear Phenomena*, vol. 183, no. 3-4, pp. 190–204, Sep. 2003.

[27] C. M. Bishop, *Pattern Recognition and Machine Learning*. Secaucus, NJ, USA: Springer-Verlag New York, Inc., 2006.